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Numerical Analysis for Mathematical Model of Heart Excitation

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Abstract

Numerical methods for mathematical model of heart excitation are considered, which is used to simulate the process of changing the field in two domains corresponding to the heart. The alteration of the potential in the heart is described by the reaction diffusion equation. They are solved by Backward Time Central Space (BTCS) scheme and Crank-Nicolson scheme. Moreover the reaction terms implicitly have good stability properties. Managing such methods can find the ways to provide suitable approximations and can compare with the other methods.

Keywords: *BTCS scheme, Crank-Nicolson scheme, Reaction-diffusion equation, Numerical analysis*

1. INTRODUCTION

Cardiac muscle is a type of automatic stringy muscle establish in the walls and basis of the heart. It is one of three major types of muscle, the other being smooth muscle and skeletal (Denisov and Kalinin, 2010). Heart failure (HF) is a chronic disease characterized by the disability of the heart to pump an enough volume of blood to reach the request of the different stalk systems. In the heart, electrical excitation propagates through diffusively coupled cardiac cells and subsequently results in contraction and force generation. Electrical cardiac cells may in turn change electrophysiological properties of the tissue, action potential duration, or induce after-depolarization resulting in premature beats. This model consists of reaction-diffusion equations describing cardiac electrophysiology, equations explaining the tissue mechanics.

Numerical simulations indicate that mechanical deformation may result in spiral wave drift and subsequent breakup. Mathematical modelling is an important role in the study of electrophysiological processes in the heart. The mathematical report of this problem on the basis of available cardiac muscle excitation models leads to the problem for evolutionary partial differential equations. It is a condition representing the end-phase of a swarm of other cardiac diseases without a curative nurture. Once diagnosed, the rest of the patients' life to improve their life quality and survival require the medication. On these days, it seems the success in treating other heart conditions like heart disease, and arrhythmias are increased due to the rash of heart failure which helps the patients have longer lives. Simulation and modelling are being used as tools for studying the cardiac electrical activity. Cardiac model, as dynamic activity equation, are used in explanation of membrane action potentials. Those models are composed of unit cells assigned with an individual action potential with different characteristics ranging from a step function to a simple rectangular function in order to describe the details of potential phases and physiological cells (Van De Vosse, 2003).

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Electrophysiological processes in the cardiac muscle are characterized by the kind of the transmembrane potential in the cellular membranes and by the generation of an electric field in the environment (Van De Vosse, 2003). A big group of monodomain models can be explained by the following problems:

$$u_t = Du_{xx} + f(u(x,t), v(x,t)), (x,t) \in \Omega \times (0,T], \quad (1.1)$$
$$v_t = F\big(u(x,t)\big), (x,t) \in \Omega \times (0,T], \quad (1.2)$$
$$u(x,0) = \emptyset(x), x \in \Omega \quad (1.3)$$
$$v(x,0) = 0, \in \Omega \quad (1.4)$$

Here, Ω is a domain in the three-dimensional space corresponding to the cardiac muscle,
$\partial\Omega$ is the boundary of Ω,
$u(x,t)$is the transmembrane potential,
$v(x,t)$is the density of the transmembrane ionic current,
D is the specific electroconductivity coefficient,
$f(u,v)$ and $F(u,v)$are the activation functions,
$\emptyset(x)$ is the transmembrane potential at the initial time of the cardiac muscle excitation,
T is the duration of the cardiac cycle.

In the research we want to use methods of partial differential equation. The analytical solutions are difficult or impossible to obtain and solutions must be approximated numerically. To approximate the model by finite differences, we divide the closed domain by a set of lines parallel to the spatial and time axes to form a grid or a mesh. We shall assume, for simplicity, that the sets of lines are equally spaced such that the distance between crossing points is Δx and Δt . Our models are calculated by Backward Time Central Space (BTCS) and Crank-Nicolson (CN) schemes. We have to compare the solutions in order find a good method.

2. PRELIMINARIES

In this part, we give some definitions, notations and some useful results that will be used. Throughout this research, we let R be the set of all real numbers and N be the set of all natural numbers.

2.1 Reaction-Diffusion equations

Reaction-diffusion (RD) equations, happened physically in systems consisting of many interacting components are normally used to describe formation phenomena of biological, chemical and physical systems. The important ingredients of all these models are in the form:

$$u_t = Du_{xx} + f(u), \quad (2.1)$$

Where,
$u = u(x,t)$ is a vector of concentration variables, $f(u)$ describes a local reaction kinetics and Ddenotes a diagonal diffusion coefficient matrix. Therefore, the system is the combination of isotropic and uniform so that D is represented by a scalar matrix (Van De Vosse, 2003). Suppose that the initial distribution $u(x,0)$ is a function of a space interval $x \in (-\infty, +\infty)$

2.2 Finite difference approximation

Let $u(x)$be a function of variable x, it will be assumed to be smooth, meaning that we can differentiate the function many times and each derivative are well defined bounded functions over an interval containing a set of points of interest x (Denisov and Kalinin, 2010). The finite difference approximation of the derivative of $u$can be written by:

$$u_x \approx \frac{u_{j+1} - u_j}{\Delta x} \quad (2.2)$$

Where, $u(x,t)$is continuous solution (exact solution). u_n^j is approximate numerical solution.

3. MAIN RESULTS

In this part, we review methods for formulating partial differential equations based on the random field representations.

The deviation between variants of monodomain models are normally in the way of activation functions $f(u,v)$and $\emptyset(u,v)$are particularized. These models are based on a strictly style of the properties of membrane ionic channels in the frameworks of the Hodgkin-Huxley theory.
In the second group of models, the functions $f(u,v)$and $\emptyset(u,v)$are specified using empirical formulas. Equations (3.1)-(3.3) describe two phases of the electrophysiological process depolarization and repolarization. If we examine of the first (depolarization) phase that is the electrophysiological processes at $t \in (0,T_1]\left(T_1 < \frac{T}{2}\right)$. We may set $F(u,v) = 0$. In this case; the model of the cardiac muscle excitation becomes simple and takes the form:

$$u_t = Du_{xx} + f(u(x,t),0), (x,t) \in \Omega \times (0,T], \tag{3.1}$$
$$u_x = 0x \in \Omega, \tag{3.2}$$
$$u(x,0) = \gamma(x), \in \Omega. \tag{3.3}$$

3.1 Backward Time Central Space scheme

The explicit schemes are simple but they are only conditionally stable. On the other hand, implicit methods is typically unconditionally stable and find a solution by solving equations involving the present state of the models and the later one (Causon and Mingham, 2010). Use the finite difference approximations with

$$u_t = Du_{xx} + f\big(u(x,t)\big) \tag{3.4}$$

We have:

$$\frac{u_{n+1}^j - u_n^j}{\Delta t} = D\,\frac{u_n^{j+1} - 2u_n^j + u_n^{j-1}}{(\Delta x)^2} \tag{3.5}$$

$$u_{n+1}^j = ru_n^{j+1} + (1-2r)u_n^j + ru_n^{j-1} + f\big(u_n^j\big)\Delta t, \tag{3.6}$$

Where, $r = \frac{D\Delta t}{(\Delta x)^2}$ and $c = 1 - 2r$.

From the Neumann boundary condition $\frac{\partial u}{\partial n}(x,t) = 0$, we can write (3.6) in by matrix from as:

$$\begin{pmatrix} u_{n+1}^1 \\ u_{n+1}^2 \\ u_{n+1}^3 \\ . \\ . \\ . \\ u_{n+1}^m \end{pmatrix} = \begin{pmatrix} c & 2r & 0 & . & . & . & 0 \\ r & c & r & . & . & . & 0 \\ 0 & r & c & 2r & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & r & c & r \\ 0 & 0 & 0 & 0 & 0 & 2r & c \end{pmatrix} \begin{pmatrix} u_n^1 \\ u_n^2 \\ u_n^3 \\ . \\ . \\ . \\ u_n^m \end{pmatrix} + \Delta t \begin{pmatrix} f(u_n^1) \\ f(u_n^2) \\ f(u_n^3) \\ . \\ . \\ . \\ f(u_n^m) \end{pmatrix}$$

3.2 Crank-Nicolson scheme

Combining the stability of the implicit method with the accuracy of a method that is second-order in space and in time that is possible and achieved by averaging explicit, Forward Time Central Space (FTCS) and implicit BTCS schemes. Combine finite difference an approximation becomes:

$$u_t = Du_{xx} + f\big(u(x,t)\big) \tag{3.7}$$

Use the finite difference approximations with Eq. (3.7), becomes

$$\frac{u_{n+1}^{j}-u_{n}^{j}}{\Delta t}=\frac{D}{2}\left(\frac{u_{n+1}^{j+1}-2u_{n}^{j}+u_{n+1}^{j-1}}{(\Delta x)^2}+\frac{u_{n}^{j+1}-2u_{n}^{j}+u_{n}^{j-1}}{(\Delta x)^2}\right)+f(u_{n}^{j}) \tag{3.8}$$

$$u_{n+1}^{j}=\frac{D\Delta t}{2}\left(\frac{u_{n+1}^{j+1}-2u_{n}^{j}+u_{n+1}^{j-1}}{(\Delta x)^2}+\frac{u_{n}^{j+1}-2u_{n}^{j}+u_{n}^{j-1}}{(\Delta x)^2}\right)+f(u_{n}^{j}) \tag{3.9}$$

Where, $r=\frac{D\Delta t}{(\Delta x)^2}$.

$$-ru_{n+1}^{j+1}+(1+2r)u_{n+1}^{j}-ru_{n+1}^{j+1}=ru_{n}^{j-1}+(1-2r)u_{n}^{j}+ru_{n}^{j+1}+f(u_{n}^{j})\Delta t, \tag{3.10}$$

We can write by metrix:

$$\begin{pmatrix} 1+2r & -r & 0 & . & . & . & 0 \\ -r & 1+2r & -r & . & . & . & 0 \\ 0 & -r & 1+2r & -r & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & -r & 1+2r & -r \\ 0 & . & . & . & 0 & -r & 1+2r \end{pmatrix} \begin{pmatrix} u_{n+1}^{1} \\ u_{n+1}^{2} \\ u_{n+1}^{3} \\ . \\ . \\ . \\ u_{n+1}^{m} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} f(u_{n}^{1}) \\ f(u_{n}^{2}) \\ f(u_{n}^{3}) \\ . \\ . \\ . \\ f(u_{n}^{m}) \end{pmatrix}+\begin{pmatrix} ru_{n}^{0}+(1-2r)u_{n}^{j}+ru_{n}^{j}+ru_{n+1}^{j} \\ ru_{n}^{j}+(1-2r)u_{n}^{j}+ru_{n}^{j} \\ ru_{n}^{j}+(1-2r)u_{n}^{j}+ru_{n}^{j} \\ . \\ . \\ . \\ ru_{n}^{j-2}+(1-2r)u_{n}^{j-1}+ru_{n}^{j}+ru_{n}^{j+1} \end{pmatrix}$$

4. EXAMPLE

In this section, there is an example to show how this method works for diffusion system. We use Eq. (3.1-3.3) to simulate model.

$$u_t = Du_{xx} + \sin(3x)\, e^{-t},$$
$$u(x,0) = \sin(x),$$
$$u(0,t) = 0,$$
$$u(L,t) = 0\,.$$

The exact solution is:

$$u(x,t) = e^{-t}\sin(x) + \tfrac{1}{8}(e^{-t} - e^{9-t})$$

The numerical solution of this mathematical model by BTCS and Cranks-Nicolson method is shown in table 4.1 and figure 4.1-4.2.

Table 4.1. The solutions of BTCS, Crank-Nicolson schemes and exact. Where, T=15.

x	BTCS	Crank-Nicolson	Exact
0.0001	0.00344484650	0.00129045	0.00010000000
0.0002	0.00117347660	0.00119292	0.00005194080
0.0003	0.00057740610	0.00084828	0.00001471550
0.0004	0.00046644430	0.00064273	0.00000370590
0.0005	0.00051560530	0.00058458	0.00000087490
0.0006	0.00060347990	0.00059895	0.00000019830
0.0007	0.00070076270	0.00061504	0.00000004370
0.0008	0.00080016330	0.00057618	0.00000000940
0.0009	0.00090003490	0.00044897	0.00000000200
0.001	0.00000000350	0.00024277	0.00000000040

Table 4.2. The error of BTCS and Crank-Nicolson schemes.

x	Absolute error of BTCS	Absolute error of Crank-Nicolson
0.0001	0.0033448465	0.0011904480
0.0002	0.0011215358	0.0011409824
0.0003	0.0005626906	0.0008335601
0.0004	0.0004627384	0.0006390281
0.0005	0.0005147303	0.0005837093
0.0006	0.0006032816	0.0005987501
0.0007	0.0007007190	0.0006149994
0.0008	0.0008001539	0.0005761750
0.0009	0.0009000329	0.0004489665
0.001	0.0000000031	0.0002427647

Table 4.1 is explained the potential in cardiac excitation. One specific period of time; the potential is decreasing. From the previous table expresses the solution of Crank-Nicolson is closer to exact solution than BTCS scheme. Table 4.2 shows the error of Crank-Nicolson scheme that get the result which better than BTCS scheme. We also possible to measure the absolute different $||x - y||$between vectors $x$and y

Moreover, we get root mean square of BTCS is 0.0013 and Crank-Nicolson is 0.0007. We can explain that both of these errors can apply to use with our model.

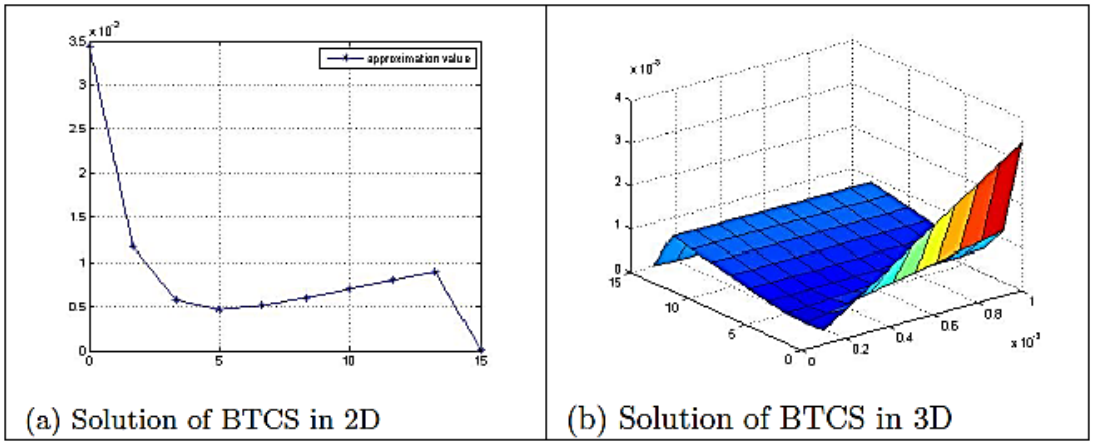

Fig 4.1 show the solutions that solved by BTCS. We show it to compare the potentials and times.

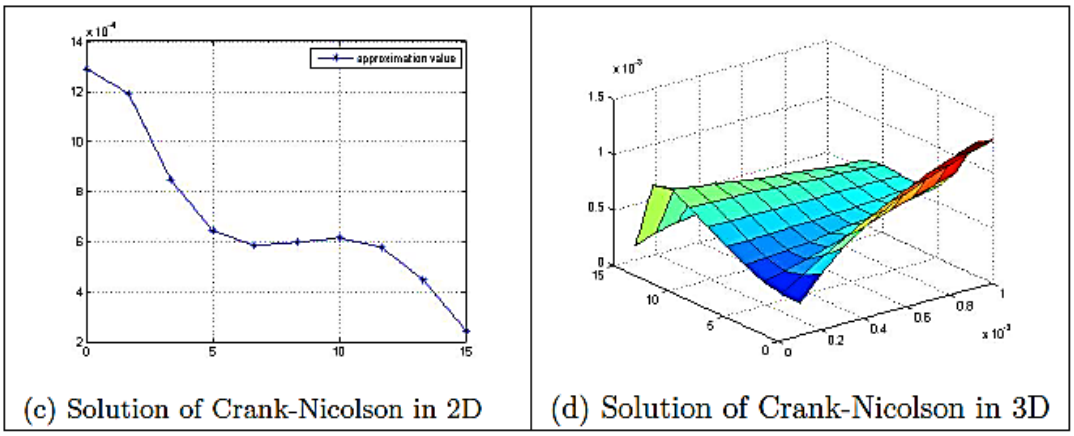

Fig 4.2 shows the solutions that solved by Crank-Nicolson. We show it to compare the potentials and times.

The system of nonlinear reaction-diffusion equations is solved by BTCS scheme and Crank-Nicolson scheme. These figures show comparison of the simulated transmembrane potential. From the model, we have to fix x in the overtime and the potential is lower. According to the ECG puts on body that means we have to fix space on body.

5. CONCLUSION

The constructed implicit finite difference scheme can be directly applied to solve the transformed (system with diffusion terms to system without diffusion terms) reaction diffusion system. After that, the solutions of the reaction diffusion system are obtained by applying inverse of the transformation. The results of the numerical experiments presented here show that the proposed numerical method for solving the reaction diffusion can use both of BTCS and Crank-Nicolson.

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